(in contrast to the second variant), the steepness and velocity of a wave affecting the stability of its evolution lie outside the wave slope vs wave velocity diagram which holds for actual waves ([1], Fig. 6.4-2). The above peculiarities in the evolution of wind waves are in agreement with the results of observations in nature.

For values in the range $0 < \mathrm{Fr} < 1/R - R$, a trend opposite to that described above prevails: The leeward wave slopes become flatter, while the windward slopes become steeper.

Capillary Wind Wave. Variant 3: $W = 3.995$; $u = -1$; $\zeta(a, 0) = a + i0.2\pi \sin a$; $\Gamma(a, 0) = 2 - 0.4\pi \sin a$. The calculations are performed for the time up to the moment $t = 1$ for the interval $\Delta t = 1/90$ without taking into account the derivatives Γ_{tt} and ζ_{ttt}. The wave evolution shown in Fig. 4 (the wave peaks and troughs are also connected by dashed straight-line segments) is similar to the evolution of gravitational waves. However, it is also characterized by the fact that the wave tops become flatter and the troughs deeper.

With a reduction in the Froude and Weber numbers in comparison with those indicated in variants 1-3, the wave evolution is retarded, while the critical interval Δt_* remains almost unchanged. The latter leads to the fact that, in calculating ripple waves, the role of the nonlinear effects caused by the finiteness of the wave amplitude is not revealed even if a large amount of computer time is used. This means that the linear theory adequately describes the motion of finite-amplitude ripple waves.

LITERATURE CITED

1. B. Kinsman, Wind Waves, Prentice-Hall, Englewood Cliffs, New Jersey (1965).
2. T. P. Barnett and K. E. Kenyon, "Recent advances in the study of wind waves," Rep. Prog. Phys., 38, No. 6 (1975).
3. R. L. Kulyaev, "Investigation of finite-amplitude internal waves," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1975).
4. H. Lamb, Hydrodynamics, 6th ed., Dover (1932).

HYDRODYNAMIC STABILITY OF TWO-DIMENSIONAL POISEUILLE FLOW OF A NON-NEWTONIAN LIQUID WITH A HIGH-VISCOSITY CORE IN A COOLED CHANNEL

A. S. Romanov

UDC 532.51:532.135

Viscoplastic liquids occupy an important position among non-Newtonian liquids [1, 2]. The hydrodynamic stability of the two-dimensional Poiseuille flow of these liquids was investigated in [3, 4]. The mechanical characteristics of viscoplastic media are determined by the dimensionless rheological equation, which relates the stress tensor deviator σ_{ij} to the strain rate tensor f_{ij} [1]:

$$\sigma_{ij} = 2\left(1 + \frac{\varkappa}{\sqrt{2 f_{ij} f_{ij}}}\right) f_{ij} \quad \text{for} \quad \sqrt{\frac{1}{2}\sigma_{ij}\sigma_{ij}} \geqslant \varkappa,$$
$$f_{ij} = 0 \qquad \text{for} \quad \sqrt{\frac{1}{2}\sigma_{ij}\sigma_{ij}} \leqslant \varkappa, \tag{1}$$

where $\varkappa = \tau_0 L/\mu U$ is the plasticity parameter; μ is the plastic dynamic viscosity; τ_0 is the ultimate shearing stress; L is the characteristic dimension (half-width of the channel); and U is the characteristic velocity. Due to the existence of the ultimate shearing stress τ_0 for a viscoplastic liquid, zones where the medium moves as a quasisolid body as well as viscous flow zones can form in the flow of such a liquid through channels [2].

The dimensionless shearing stress τ as a function of the dimensionless shearing rate δ for unidimensional shear flow of a viscoplastic liquid (1) is shown in Fig. 1. The rheological equation (1) is approximate for many actual liquids and the flow curve is essentially nonlinear for low shearing rates [5] (dashed curve in Fig. 1). The rheological law is in this case written conveniently as

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 86-90, November-December, 1977. Original article submitted October 18, 1976.

$$\sigma_{ij} = 2\eta(\omega) f_{ij}, \tag{2}$$

where $\omega \equiv \sqrt{2f_{ij}f_{ij}}$; $\eta(\omega)$ is a continuous monotonic function, which is bounded for $0 \le \omega \le \infty$; $1 \le \eta(\omega) \le \eta(0)$. If the intensity of the deformation-rate tensor ω is larger than, or equal to, a certain value $\omega \ge \omega^* \ll 1$, the rheological law (2) coincides with (1). The flow regions where $\omega < \omega^*$ will be referred to as high-viscosity-flow zones.

Experiments show that the non-Newtonian characteristics of many viscoplastic media depend considerably on the temperature [6]. At a certain temperature $T = T^*$ they vanish altogether ($\tau_0 = 0$ for $T \ge T^*$). Therefore, we shall subsequently consider that $\tau_0 = \tau_0(T)$ and $\eta = \eta(T, \omega)$.

The symmetric distribution of the liquid velocity in a two-dimensional channel which is cooled symmetrically with respect to its axis under the action of a constant dimensionless pressure gradient $\partial p/\partial x = -2/\mathrm{Re}$ ($\mathrm{Re} = \rho UL/\mu$ is the Reynolds number; ρ is the density of the medium) under the condition that the flow parameters vary little along the channel ($\partial/\partial x \approx 0$) is determined by

$$u(y) = \begin{cases} (1-\zeta)^2 + \int\limits_{-\zeta}^{-1} \varkappa\, dy + \varkappa(-\zeta)(1-\zeta) + O(\omega^*) & \text{for } -\zeta \leqslant y \leqslant 0, \\ y^2 - 2\zeta y + 1 - 2\zeta + \int\limits_{y}^{-1} \varkappa\, dy + \varkappa(-\zeta)(1+y) + O(\omega^*) & \\ \qquad \text{for } -1 \leqslant y \leqslant -\zeta, \end{cases} \tag{3}$$

where $\zeta = \varkappa/2 + O(\omega^*)$ is the half-width of the high-viscosity-flow core, which is determined from the equilibrium condition, while $\omega(-\zeta) = \omega^* \ll 1$. The velocity distribution (3) holds only if $\varkappa[T(y)]|_{y=-1} \leqslant 2$, since, otherwise, a high-viscosity zone forms at the channel walls, and distribution (3) does not hold.

Consider the hydrodynamic stability of the flow (3) with respect to infinitesimal, two-dimensional, and periodic with respect to time and channel length perturbations of the stream function

$$\psi(x, y, t) = \varphi(y) \exp i\alpha(x - ct)$$

and the temperature

$$\theta(x, y, t) = \gamma(y) \exp i\alpha(x - ct),$$

where α and αc are the dimensionless wave number and the complex perturbation frequency, respectively. Although the inapplicability of Squire's theorem in the general case of non-Newtonian media has been demonstrated in [7], the problem of the effect of three-dimensionality of the perturbing motion on the hydrodynamic stability of two-dimensional gradient flow of a non-Newtonian liquid characterized by the rheological law (2) and a function $\eta = \eta(\omega)$ of arbitrary type should be the subject of an independent investigation.

We obtain the following from the system of equations of motion and energy, neglecting the evolution of heat due to viscous dissipation, with an accuracy to the infinitesimal values of the higher orders for $\varphi(y)$ and $\gamma(y)$:

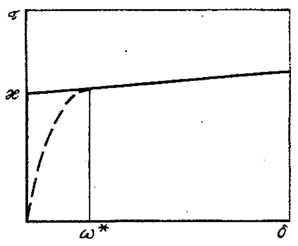

Fig. 1

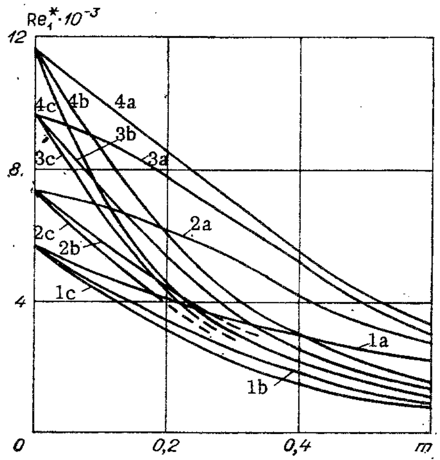

Fig. 2

$$(u-c)\gamma + T_0'\varphi = \frac{1}{i\alpha \operatorname{Re} \Pr}(\gamma'' - \alpha^2\gamma); \tag{4}$$

$$\begin{aligned}(u-c)(\varphi''-\alpha^2\varphi) - u''\varphi = \frac{1}{i\alpha \operatorname{Re}}\{\eta_0(\varphi^{\text{IV}} - 2\alpha^2\varphi'' + \\ + \alpha^4\varphi) + 2\eta_0'(\varphi''' - \alpha^2\varphi') + \eta_0''(\varphi'' + \alpha^2\varphi) + \\ + \left[u'\left(\frac{\partial\eta}{\partial\omega}\right)_0(\varphi''+\alpha^2\varphi) + u'\left(\frac{\partial\eta}{\partial T}\right)_0\gamma\right]'' + \\ + u'\alpha^2\left[\left(\frac{\partial\eta}{\partial\omega}\right)_0(\varphi''+\alpha^2\varphi) + \left(\frac{\partial\eta}{\partial T}\right)_0\gamma\right]\Big\},\end{aligned} \tag{5}$$

where $\Pr = \mu C/\lambda$ is the Prandtl number; C is the specific heat of the medium; λ is the thermal conductivity coefficient; the zero subscript denotes unperturbed flow; the primes denote differentiation with respect to y.

Equation (5) is simplified in the range $-1 \le y \le -\xi$:

$$(u-c)(\varphi''-\alpha^2\varphi) - u''\varphi = \frac{1}{i\alpha \operatorname{Re}}(\varphi^{\text{IV}} - 2\alpha^2\varphi'' + \alpha^4\varphi) + \frac{4}{i\alpha \operatorname{Re}}\left[\left(\frac{d\varkappa_0}{dT}\gamma\right)' + \alpha^2\left(\frac{d\varkappa}{dT}\right)_0\gamma - \alpha^2\left(\frac{\varkappa}{u'}\varphi'\right)'\right]. \tag{6}$$

Development of hydrodynamic instability in the flow is related to the presence in the channel of a critical viscous layer with a thickness on the order of $(\alpha\operatorname{Re})^{-1/3}$ around the critical point $y = y_c$, $u(y_c) = \text{Real } c$ [8]. After the substitution of the independent variable $z = (y - y_c)\varepsilon^{-1}$, $\varepsilon = [\alpha\operatorname{Re}\, u'(y_c)]^{-1/3}$ in the critical layer region, Eqs. (4) and (6) assume the following form:

$$-\frac{1}{u'(y_c)}\left[\left(\frac{dT_0}{dy}\right)_c + O(\varepsilon)\right]\varphi + \varepsilon\left(\gamma - \frac{1}{i\Pr}\frac{d^2\gamma}{dz^2}\right) = O(\varepsilon^2), \tag{7}$$

$$\frac{d^4\varphi}{dz^4} - iz\frac{d^2\varphi}{dz^2} + 4\varepsilon^3\frac{d}{dz}\left[\left(\frac{d\varkappa_0}{dT}\right)_c\gamma\right] = O(\varepsilon).$$

It follows from Eqs. (7) that, if $\Pr \ll \varepsilon^{-2}(d\varkappa_0)/(dy)_c^{-1}$, the velocity perturbations can be considered as being independent of temperature perturbations, while the subscript c means that the quantity in question is calculated at the critical point $y = y_c$. If this condition is satisfied, the perturbation of the stream function for $-1 \le y \le -\xi$ is determined from Eq. (6) in the following form:

$$\varphi = \sum_{k=1}^{4} C_k\varphi_k,$$

where $\varphi_{1,2}$ and $\varphi_{3,4}$, the "nonviscous" and "viscous" integrals of Eq. (6) [7, 9], are written thus:

$$\varphi_1 = \sum_{k=0}^{\infty} a_k(y-y_c)^{k+1}, \quad \varphi_2 = 2a_1\varphi_1 \ln(y-y_c) + \sum_{k=0}^{\infty} b_k(y-y_c)^k,$$

$$\varphi_{3,4} = \int_{\pm\infty}^{z} dz \int_{\pm\infty}^{z} \sqrt{z}H_{1/3}^{(1,2)}\left[\frac{2}{3}(iz)^{2/3}\right]dz.$$

Here $H_{1/3}^{(1,2)}$ are Hankel functions of order 1/3 of the first and the second kind.

In the region of the high-viscosity core $-\xi \le y \le 0$, the general solution for perturbations of the stream function can be obtained directly from Eq. (5),

$$\varphi^{(\text{I})} = \sum_{k=1}^{4} C_k^{(\text{I})}\varphi_k^{(\text{I})},$$

where the values of $\varphi_k^{(\text{I})}$, under the assumption that $[\eta(0)(\alpha\operatorname{Re})^{-1}]^{1/2} \ll 1$, are found from Eq. (5) in the following form:

$$\varphi_{1,2}^{(\text{I})} = \exp(\pm\alpha y) + O[(\alpha\operatorname{Re})^{-1}],$$

$$\varphi_{3,4}^{(\text{I})} = \exp\left\{\pm\sqrt{i\alpha\operatorname{Re}}\int_0^y \sqrt{(u-c)\left[\eta_0 + u'\left(\frac{\partial\eta}{\partial\omega}\right)_0\right]}\,dy\right\}\{1 + O[(\alpha\operatorname{Re})^{-1/2}]\}.$$

The boundary conditions for perturbations of the stream function are the conditions of "adhesion" of the liquid to the walls and the conditions of symmetry relative to the channel axis:

$$\varphi(-1) = \varphi'(-1) = \varphi'(0) = \varphi'''(0) = 0. \tag{8}$$

The condition for the nontriviality of the solutions φ and $\varphi^{(I)}$, the boundary conditions (8), and the joining conditions for the solutions φ and $\varphi^{(I)}$ for $y=y_c$,

$$\frac{d^k\varphi}{dy^k}=\frac{d^k\varphi^{(I)}}{dy^k},\quad k=0,1,2,3$$

[with an accuracy to quantities of the order of $(\alpha \mathrm{Re})^{-1/2}$], lead to a secular equation for determining the eigenvalues of the phase velocity of perturbation

$$\frac{\varphi_3(-1)}{\varphi_3'(-1)}=\frac{\varphi_2(-1)\left[\varphi_1'(-\zeta)+\alpha\varphi_1(-\zeta)\,\mathrm{th}\,\alpha\zeta\right]-\varphi_1(-1)\left[\varphi_2'(-\zeta)+\alpha\varphi_2(-\zeta)\,\mathrm{th}\,\alpha\zeta\right]}{\varphi_2'(-1)\left[\varphi_1'(-\zeta)+\alpha\varphi_1(-\zeta)\,\mathrm{th}\,\alpha\zeta\right]-\varphi_1'(-1)\left[\varphi_2'(-\zeta)+\alpha\varphi_2(-\zeta)\,\mathrm{th}\,\alpha\zeta\right]}. \tag{9}$$

It follows from (9) that, in the above approximation, the stability is affected by the velocity distribution $u=u(y)$ only in the region $-1\le y\le -\zeta$; the velocity distribution depends on the function $\varkappa=\varkappa[T(y)]$, which determines the variation of the non-Newtonian characteristics of the medium.

As an example, we shall assign the function $\varkappa=\varkappa[T(y)]$ in the variation interval of the independent variable $y[-1;-\zeta]$ in the form $\varkappa=2\zeta+m(1-\zeta)^{1-n}(-y-\zeta)^n$, which approximates the variation of non-Newtonian properties of the medium across a channel with cooled walls.

Figure 2 shows the dependence of the critical Reynolds number Re_1^* $[\mathrm{Re}_1=\mathrm{Re}/(1-\zeta)^3]$ on the parameters m and n, which characterize the variability of the non-Newtonian properties of the medium across the channel for different dimensions of the high-viscosity flow core ζ. The curves 1-4 correspond to $\zeta=0$, 0.3, 0.6, 0.9, while the curves marked by letters a, b, and c correspond to parameter values $n=2$, 3, 4. Analysis of the data obtained shows that cooling of the walls greatly destabilizes the non-Newtonian liquid flow (2). This fact is qualitatively supported by the well-known experimental data obtained for viscoplastic petroleum [10].

The author is grateful to K. B. Pavlov for the discussion.

LITERATURE CITED

1. Z. P. Shul'man, Convective Heat and Mass Transport of Rheologically Complex Liquids [in Russian], Énergiya, Moscow (1975).
2. P. M. Ogibalov and A. Kh. Mirzadzhanzade, Nonstationary Motion of Viscoplastic Media [in Russian], Izd. Mosk. Univ., Moscow (1970).
3. K. B. Pavlov, A. S. Romanov, and S. L. Simkhovich, "Hydrodynamic stability of the Poiseuille flow of a non-Newtonian viscoplastic liquid," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1974).
4. K. B. Pavlov, A. S. Romanov, and S. L. Simkhovich, "Stability of the Poiseuille flow of a viscoplastic liquid with respect to finite-amplitude perturbations," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1975).
5. W. Philipoff, "Zur Theorie der Strukturviskosität," Kolloid Z., No. 71 (1935).
6. V. E. Gubin, Yu. V. Skripnikov, and L. S. Abramzon, "Static shearing stress in viscoplastic oils," Tr. Ufim. Nauchno-Issled. Inst. Trans. Neft., No. 7 (1970).
7. A. T. Listrov, "Stability of parallel flows of non-Newtonian media," Dokl. Akad. Nauk SSSR, 164, No. 5 (1965).
8. Lin Chia-Ch'ias, Hydrodynamic Stability, Cambridge University Press.
9. W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Wiley-Interscience (1966).
10. L. S. Abramzon and V. A. Belozerov, Method of Calculating "Hot" Piping for the Steady-State Pumping of High-Congealing Oil and Oil Products [in Russian], VNIIOÉNG, Moscow (1970).